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LETTER TO THE EDITOR

r-adic one-dimensional maps and the Euler summation formula

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Abstract. For the r-adic one-dimensional maps, we explicitly construct the decaying eigenstates and adjoint eigenstates associated with the Ruelle resonances. We show that the eigenfunctions of the corresponding Frobenius–Perron operator are the well known Bernoulli polynomials. The adjoint eigendistributions are obtained as derivatives of the Dirac distributions at the end points of the unit interval. The resulting expansion of the initial density in terms of the decaying eigenstates is given by the Euler summation formula.

Several recent works have been devoted to Ruelle's resonances [1] in chaotic dynamical systems [1–11]. The Ruelle resonances are the poles of the spectral functions which are the Fourier transforms of the time correlation functions [2–4]. These resonances are calculated by solving the eigenvalue problem of the Frobenius–Perron operator corresponding to the dynamical system [12]. The resonances are obtained as the zeros of the Fredholm determinant of the evolution operator which, in turn, can be written as a product over the unstable periodic orbits of the chaotic system [4–7]. This method has been shown to be very powerful to obtain the relaxation or the decay rates of several chaotic systems such as one-dimensional maps, the disk scatterers, and others [4–8].

However, the nature of the eigenstates which are associated with these Ruelle resonances remains elusive [1, 9–11]. In this letter, we construct these eigenstates for a family of exactly solvable systems which are the r-adic one-dimensional maps [13]

$$x_{t+1}=\varphi(x_t) \qquad \text{with} \qquad \varphi(x)\equiv rx \qquad (\text{modulo } 1) \tag{1}$$

where r is an integer greater than or equal to 2. We associate two operators with the dynamical system (1). One one hand, we have the Koopman operator $(\hat{U}g)(x)\equiv g(\varphi(x))$ acting on functions $g(x)$ defined in the unit interval [12]. On the other hand, we have the Frobenius–Perron operator [12]

$$(\hat{P}f)(x)\equiv \sum_{\tilde{x}:\, x=\varphi(\tilde{x})} \frac{f(\tilde{x})}{|\varphi'(\tilde{x})|}=\frac{1}{r}\sum_{m=0}^{r-1} f\left(\frac{x+m}{r}\right) \tag{2}$$

which is related to the Koopman operator in the well known manner [12].

We asume that the Frobenius–Perron operator acts on the complete countably normed (Fréchet) space [15] $\mathscr{E}(\tau)$ of analytic entire functions f of exponential type τ, i.e. $\forall\varepsilon>0$, $\exists A_\varepsilon(f)>0$:

$$|f(z)|\leqslant A_\varepsilon \exp[(\tau+\varepsilon)|z|] \qquad \forall z=x+\mathrm{i}y\in\mathbb{C} \tag{3}$$

where the topology is given by the countable family of norms [16]

$$\|f\|_p\equiv \sup_{z\in\mathbb{C}} \exp\left[-\left(\tau+\frac{1}{p}\right)|z|\right]|f(z)| \qquad p\in\mathbb{N}_0. \tag{4}$$

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We observe [1] that the entire functions $\{\cos 2\pi lx, \sin 2\pi lx\}$ with $l \in 2\mathbb{N}+1$ are mapped onto the zero function by $\hat{P}$ so that they are eigenfunctions associated with the eigenvalue $\chi = 0$ of $\hat{P}$. In order to remove this null space of $\hat{P}$, we shall fix the parameter τ at a value less than 2π.

A basis of $\mathscr{E}(\tau)$ is given by the monomials $\{x^k\}_{k\in\mathbb{N}}$ in which the Frobenius–Perron operator becomes an infinite triangular matrix with the diagonal elements

$$\chi_n = \frac{1}{r^n} \qquad n = 0, 1, 2, \ldots. \tag{5}$$

Therefore, the Fredholm determinant of $\hat{P}$ is given by

$$0 = \det(I - \chi^{-1}\hat{P}) = \prod_{n=0}^{\infty}\left(1 - \frac{1}{\chi r^n}\right). \tag{6}$$

Whereupon, we obtain the Ruelle resonances, i.e. the eigenvalues of $\hat{P}$, as the roots (5) of the characteristic determinant (6).

The eigenfunctions of the Frobenius–Perron operator associated with the resonances (5) are the Bernoulli polynomials $B_n(x)$ in the sense that

$$(\hat{P}B_n)(x) = \frac{1}{r^n} B_n(x). \tag{7}$$

This fact can easily be proved by applying $\hat{P}$ on the generating function of the Bernoulli polynomials given in [17] (see also [18, 19]).

The dynamical problem is the following. We are interested in the time evolution of an ensemble of particles starting from the initial density $f(x)$. According to the Euler summation formula (23.1.32 in [17]), we have

$$f(x) = \int_0^1 f(y)\,\mathrm{d}y + \sum_{n=1}^{\infty} \frac{1}{n!}[f^{(n-1)}(1) - f^{(n-1)}(0)]B_n(x) \tag{8}$$

where $f^{(k)}$ denotes the kth derivative of f. From the properties of the Bernoulli polynomials [17], the right-hand member of (8) is convergent if f belongs to $\mathscr{E}(\tau)$ with $\tau < 2\pi$ which confirms the critical value for τ. Methods to overcome this restriction have been discussed in [20]. For numerical applications, it is known that excellent results can be obtained by truncating the divergent series (8) while keeping the rest. In this respect, the spectral decomposition (8) differs from standard L^2 spectral decompositions [14].

Equation (8) implies that the eigendistributions associated with the eigenfunctions (7) are

$$D_0(x) = \theta(x)\theta(1-x) \tag{9}$$

$$D_n(x) = \frac{(-1)^{n-1}}{n!}[\delta^{(n-1)}(x-1) - \delta^{(n-1)}(x)] \tag{10}$$

where $\theta(x)$ is the Heaviside step function while $\delta^{(k)}(x)$ denotes the kth derivative of the Dirac distribution.

The time evolution of the density can now be written in terms of Ruelle's resonances (5) and its associated decaying eigenstates

$$f_t(x) = (\hat{P}^t f_0)(x) = \sum_{n=0}^{\infty} \chi_n^t \langle D_n f_0\rangle B_n(x) \tag{11}$$

where $\langle D_n f_0 \rangle$ denotes the evaluation of the eigendistributions (9), (10) for the initial density f_0. Like the Euler summation formula (8), the expansion (11) is convergent in $\mathscr{C}(\tau)$ with $\tau < 2\pi$ but also for larger values of τ as t increases. Otherwise, it is divergent and requires a truncation before any numerical calculation. Similar results have been obtained in a work by Professor G Roepstorff [19], which came to our knowledge at the end of the present work. Let us also mention here that a different but related convergent expansion of $\hat{P}$ has been obtained (see e.g. [21]).

It is well known that the transient density relaxes towards the stationary density which is uniform for the r-adic map [12, 22]. The interesting feature of the result (11) is that the relaxation process is expressed explicitly in terms of its eigenmodes. Expansions like (11) contain more information for numerical calculations than provided with the standard L^2 spectral theory in the case of systems with a continuous spectrum on the unit circle. We believe that the methods we developed here can be extended to more general hyperbolic maps of the interval.

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References

[1] Ruelle D 1986 *Phys. Rev. Lett.* **56** 405; 1986 *J. Stat. Phys.* **44** 281; 1976 *Invent. Math.* **34** 231; 1989 *Commun. Math. Phys.* **125** 239; 1990 *IHES Publ. Math.* **72** 175

[2] Isola S 1988 *Commun. Math. Phys.* **116** 343

[3] Baladi V, Eckmann J-P and Ruelle D 1989 *Nonlinearity* **2** 119

[4] Christiansen F, Paladin G and Rugh H H 1990 *Phys. Rev. Lett.* **65** 2087

[5] Artuso R, Aurell E and Cvitanović P 1990 *Nonlinearity* **3** 325, 361

[6] Cvitanović P and Eckhardt B 1991 *J. Phys. A: Math. Gen.* **24** L237

[7] Rice S A, Gaspard P and Nakamura K 1991 *Adv. Class. Traj. Meth.* **1** 215

[8] Gaspard P 1991 Diffusion, effusion, and chaotic scattering: an exactly solvable Liouvillian dynamics *Preprint*
Gaspard P and Alonso Ramirez D 1992 Ruelle classical resonances and dynamical chaos: The three- and four-disk scatterers *Preprint*

[9] Hasegawa H and Saphir W 1992 *Phys. Lett.* **161A** 471, 477

[10] Mayer D and Roepstorff G 1987 *J. Stat. Phys.* **47** 149; 1988 *J. Stat. Phys.* **50** 331

[11] Levin G M, Sodin M L and Yuditski 1991 *Commun. Math. Phys.* **141** 119

[12] Grossmann S and Thomae S 1977 *Z. Naturforsch.* a **32** 13533
Lasota A and Mackey M C 1985 *Probabilistic Properties of Deterministic Systems* (Cambridge: Cambridge University Press)

[13] Billingsley P 1965 *Ergodic Theory and Information* (New York: Wiley)

[14] Koopman B O 1931 *Proc. Natl. Acad. Sci. USA* **17** 315
Arnold V I and Avez A 1968 *Ergodic Problems of Classical Mechanics* (New York: Benjamin)

[15] Gelfand I and Shilov G 1968 *Generalized Functions* vol 2 (New York: Academic)

[16] Lami Dozo E private communication

[17] Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions* (New York: Dover) ch 23

[18] Dörfle M 1985 *J. Stat. Phys.* **40** 93

[19] Roepstorff G 1987 unpublished

[20] Boas R P and Creighton Buck R 1958 *Polynomial Expansions of Analytic Functions* (Berlin: Springer)

[21] Mayer D 1991 *Ergodic Theory, Symbolic Dynamics, and Hyperbolic Spaces* ed T Bedford, M Keane and C Series (Oxford: Science Publication)

[22] Keller G 1984 *Commun. Math. Phys.* **96** 181